

Statistical entropy of a BTZ black hole in topologically massive gravity

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The entropy of a Bañados, Teitelboim, and Zanelli black hole in topologically massive gravity had been given with the form inconsistent with the Bekenstein-Hawking entropy. In the paper, we provide a consistent statistical interpretation for the entropy, and confirm it to be thermodynamic entropy by preserving the first and second laws. As a novel extension, the logarithmic correction to the black hole entropy is also discussed.

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I. INTRODUCTION

The black-hole solution of three-dimensional (3D) Einstein gravity with a negative cosmological constant was found by Bañados, Teitelboim, and Zanelli (BTZ) [1], and its metric takes the form

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \quad (1)$$

where ϕ is an angle with the period 2π as the identification of the black-hole spacetime. The functions N^2 and N^ϕ are

$$N^2 = -8Gm + \frac{r^2}{\ell^2} + \frac{16G^2 j^2}{r^2}, \quad N^\phi = \frac{4Gj}{r^2}, \quad (2)$$

where ℓ is the Anti-de Sitter (AdS) radius and G the 3D Newton constant. Its Killing horizons are found by setting $N^2 = 0$; this gives

$$r_{\pm} = \sqrt{2G\ell(\ell m + j)} \pm \sqrt{2G\ell(\ell m - j)}. \quad (3)$$

We may assume without loss of generality that $j \geq 0$ and assume that $\ell m \geq j$, to ensure the existence of an event horizon at $r = r_+$.

The recognition of the parameters (m, j) by conserved quantities of a black hole is significant for the black-hole thermodynamics [2], in particular for the black-hole entropy. For example, in the normal 3D Einstein gravity, the parameters (m, j) can be interpreted as the mass M and the angular momentum J of the black hole in units for which $\hbar = 1$; i.e.,

$$M = m, \quad J = j. \quad (4)$$

Here the entropy of the BTZ black hole, that is 1/4 of the length of the event horizon in Planck units, agrees with the Bekenstein-Hawking formula [3,4]. In particular, there is some understanding of the microscopic degrees of freedom [5,6] responsible for this entropy in terms of a boundary conformal field theory (CFT) and an application of the Cardy formula [7], for which the central charge is known in the semiclassical limit in which it is large [8].

In the exotic 3D Einstein gravity [9,10], however, the parameters (m, j) can be reinterpreted as the following forms,

$$M_E = \frac{j}{\ell}, \quad J_E = \ell m, \quad (5)$$

which have the reversed roles for the mass and angular momentum, compared with the case of the normal Einstein gravity. Thus in order to preserve the thermodynamic laws, an entropy, proportional to the length of the inner horizon instead of the event horizon, has to be distributed to the black hole with the same metric (1) but with Eq. (5) for the interpretation of the parameters. Usually, we called such black holes ‘‘exotic.’’ Recently, it was shown [10] that a statistical interpretation still exists in terms of a boundary CFT but with the use of a modified Cardy formula.

It is well known that both the normal and exotic 3D Einstein gravity theories propagate no physical modes (both of them have no local degrees of freedom), and the first successful attempt to generalize the 3D gravity theory to propagate the gravitons is the topologically massive gravity (TMG) [11] which is gotten by incorporating the local Chern-Simons (LCS) term into the normal 3D Einstein gravity. It is interesting to note that the BTZ metric (1) also solves the field equations of TMG, but the parameters (m, j) are related to the mass and angular momentum with the forms,

$$M_T = m + \frac{1}{\mu\ell^2} j, \quad J_T = j + \frac{1}{\mu} m, \quad (6)$$

where μ is the coupling parameter in TMG to modulate the LCS term of the theory. The forms (6) constrains the black-hole entropy by fixing it exclusively with the expression, $S_T = \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G}$, consistent with the thermodynamical first law [10], but *not* consistent with the Bekenstein-Hawking formula since a term about the length of inner horizon is included in the entropy expression. The same entropy S_T was also obtained by different methods [12–16]. However, whether such entropy still has a statistical interpretation remains unclear. In particular, a recent attempt [17] required to modify the form of the entropy S_T for some

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ranges of the coupling parameter, which will lead to the inconsistency with the thermodynamic first law. On the other hand, even though there is a statistical interpretation for the entropy S_T , it is still unclear what will provide the interpretation for it including all the ranges of the coupling parameter μ . Maybe there is a speculation or an expectation that the Cardy formula plus a modified Cardy formula used for the case of an exotic black hole [10] might provide the required statistical interpretation. But whether this is true still needs to be investigated, which is the main purpose of the paper. Moreover, for the case of TMG, not all the properties of the BTZ black-hole entropy can derive from the understanding for the cases of normal and exotic black-hole entropies, as seen by a recent revealed property [18] related to TMG, that is, the product of the areas of the inner and outer horizons is dependent on the mass. This also stimulates us to investigate the statistical interpretation for the entropy S_T and see if there is any novel results, compared with the cases of normal and exotic gravity theories.

In the paper, we will first investigate the Cardy formula, in particular the modified formula, with a fundamental method in CFT without recourse to the thermodynamic relation $E = -\partial \ln Z / \partial \beta$ (β is the reciprocal of the temperature) as was done earlier [10]. Moreover, in our investigation of Cardy formula, the leading logarithmic correction [19–22] to the entropy of a BTZ black hole appeared, and this correction was usually considered as universal when calculating the black-hole entropy using quantum theories of gravity (see Ref. [23] for general discussion about this). So in the paper, as an interesting extension, we will also investigate whether there are any influences of the LCS term on the logarithmic correction of BTZ black-hole entropy, since the leading term of the entropy has changed due to the presence of the LCS term.

The structure of the paper is as follows. We will first revisit the calculation of the Cardy formula and especially pay attention to the case with a negative central charge in Sec. II. In Sec. III we will calculate the BTZ black-hole entropy in TMG using the general Cardy formulas and discuss the thermodynamical laws. Section IV investigates the influence of LCS term on the logarithmic correction of the BTZ black-hole entropy. Finally, we discuss and summarize our results in Sec. V.

II. CARDY FORMULA

As stated above, an interpretation of a 3D BTZ black-hole entropy has to be made by the Cardy formula in the dual CFT which is the quantum counterpart of the asymptotic symmetry of asymptotically AdS space. In this section, we will focus on the investigation of modified Cardy formulas along the line of Refs. [19,20].

Start with a two-dimensional CFT, in which the central charge c can be identified by the canonical analysis of the bulk theory according to the Brown and Henneaux [8]. Here we will not review the process of obtaining the central

charge while focus on the calculation of the Cardy formula to get the BTZ black-hole entropy. The Virasoro algebra in the CFT are written as

$$\begin{aligned} [\hat{L}_m^+, \hat{L}_n^+] &= (m-n)\hat{L}_{m+n}^+ + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0} \\ [\hat{L}_m^-, \hat{L}_n^-] &= (m-n)\hat{L}_{m+n}^- + \frac{c_R}{12}m(m^2-1)\delta_{m+n,0} \\ [\hat{L}_m^+, \hat{L}_n^-] &= 0. \end{aligned} \quad (7)$$

From the Cardy calculation [7], we have the quantity

$$Z_0(\tau, \bar{\tau}) = \text{Tr} e^{2\pi i(\hat{L}_0^+ - \frac{c_L}{24})\tau} e^{-2\pi i(\hat{L}_0^- - \frac{c_R}{24})\bar{\tau}} \quad (8)$$

modular invariant in the transformation $\tau \rightarrow -\frac{1}{\tau}$. Then the partition function on a torus of modulus τ is defined as

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \text{Tr} e^{2\pi i\tau\hat{L}_0^+} e^{-2\pi i\bar{\tau}\hat{L}_0^-} \\ &= \sum \rho(L_0^+, L_0^-) e^{2\pi i\tau L_0^+} e^{-2\pi i\bar{\tau} L_0^-}, \end{aligned} \quad (9)$$

where the signs L_0^+, L_0^- without hats mean the eigenvalues of their corresponding operator matrix \hat{L}_0^+, \hat{L}_0^- . In Ref. [20], the author considered τ and $\bar{\tau}$ as independent variables and then calculated the density-of-state for the left movers with the positive central charge, i.e., $\rho(L_0^+) \approx (\frac{c_L}{96(L_0^+)^3})^{\frac{1}{2}} \exp[2\pi\sqrt{\frac{1}{6}c_L L_0^+}]$. If the central charge for the right mover is still positive, the same process led to the Cardy formula by taking the logarithm of the exponential term of the density of states; that is

$$S = 2\pi \left(\sqrt{\frac{1}{6}c_L L_0^+} + \sqrt{\frac{1}{6}c_R L_0^-} \right). \quad (10)$$

Here we calculate the case with the negative central charge and assume without loss of generality that $c_R < 0$ and $L_0^- \leq 0$. The density of states can be expressed as

$$\rho(L_0^-) = \int d\tau e^{2\pi i\tau L_0^-} Z(\bar{\tau}). \quad (11)$$

Then, according to the relation $Z(\bar{\tau}) = e^{-\frac{2\pi i c_R}{24}\bar{\tau}} Z_0(\bar{\tau})$ and the modular invariance of $Z_0(\bar{\tau})$, the density of states becomes

$$\rho(L_0^-) = \int d\bar{\tau} e^{2\pi i\bar{\tau} L_0^-} e^{-\frac{2\pi i c_R}{24}\bar{\tau}} e^{-\frac{2\pi i c_R}{24}\frac{1}{\bar{\tau}}} Z\left(-\frac{1}{\bar{\tau}}\right). \quad (12)$$

In the following we will use the saddle point approximation to calculate the integral. Define $f(\bar{\tau}) = 2\pi i\bar{\tau} L_0^- - \frac{2\pi i c_R}{24}\bar{\tau} - \frac{2\pi i c_R}{24}\frac{1}{\bar{\tau}}$, and at the extremum of $f(\bar{\tau})$, $Z(-\frac{1}{\bar{\tau}})$ varies slowly, which had been checked in Ref. [19], and thus we have

$$\rho(L_0^-) \approx \int d\bar{\tau} e^{f(\bar{\tau})} \approx \left(-\frac{2\pi}{\frac{d^2 f}{d\bar{\tau}^2} \Big|_{\bar{\tau}_0}} \right)^{\frac{1}{2}} e^{f(\bar{\tau}_0)}, \quad (13)$$

if $\bar{\tau}_0$ is a maximum and $\frac{d^2 f}{d\bar{\tau}^2}|_{\bar{\tau}_0}$ is negative. As seen, in an approximate extension of the function $f(\bar{\tau}) \simeq f(\bar{\tau}_0) + \frac{(\bar{\tau}-\bar{\tau}_0)^2}{2} \frac{d^2 f}{d\bar{\tau}^2}|_{\bar{\tau}_0}$ at its extremum, the right-hand side of (13) is explicitly computed by recognizing that the kernel of the integral is the same as the kernel of a normal density with mean $\bar{\tau}_0$ and variance $-\frac{1}{\frac{d^2 f}{d\bar{\tau}^2}|_{\bar{\tau}_0}}$. So the problem transfers into calculating the extremum of the $f(\bar{\tau})$,

$$\frac{df}{d\bar{\tau}} = 2\pi i L_0^- - \frac{2\pi i c_R}{24} + \frac{2\pi i c_R}{24} \frac{1}{\bar{\tau}^2} = 0, \quad (14)$$

and for $|L_0^-| \gg |c_R|$, we obtain

$$\bar{\tau}_0 = -i \sqrt{\frac{c_R}{24L_0^-}}. \quad (15)$$

Then substituting the result of the Eq. (15) into the definition of f function, we have

$$\begin{aligned} f(\bar{\tau}_0) &= 2\pi L_0^- \sqrt{\frac{c_R}{24L_0^-}} + \frac{2\pi c_R}{24} \sqrt{\frac{c_R}{24L_0^-}} + \frac{2\pi c_R}{24} \sqrt{\frac{24L_0^-}{c_R}} \\ &\approx 2\pi \left(L_0^- \sqrt{\frac{c_R}{24L_0^-}} + \frac{c_R}{24} \sqrt{\frac{24L_0^-}{c_R}} \right) \\ &\approx 2\pi \left(-\sqrt{\frac{|L_0^-|^2 c_R}{24L_0^-}} - \sqrt{\frac{|c_R|^2 L_0^-}{24c_R}} \right) \\ &\approx -2\pi \sqrt{\frac{1}{6} c_R L_0^-}, \end{aligned} \quad (16)$$

where the negative sign in the third line is due to $c_R < 0$, $L_0^- \leq 0$ [24]. Then using the result (13), we have

$$\rho(L_0^-) \approx \left(\frac{c_R}{96(L_0^-)^3} \right)^{\frac{1}{4}} \exp \left[-2\pi \sqrt{\frac{1}{6} c_R L_0^-} \right]. \quad (17)$$

So if $c_L > 0$ and $L_0^+ \geq 0$, we get the complete density of states as

$$\begin{aligned} \rho(L_0^+, L_0^-) &\approx \left(\frac{c_L c_R}{96^2 (L_0^+ L_0^-)^3} \right)^{\frac{1}{4}} \\ &\times \exp \left[2\pi \left(\sqrt{\frac{1}{6} c_L L_0^+} - \sqrt{\frac{1}{6} c_R L_0^-} \right) \right], \end{aligned} \quad (18)$$

the exponential term of which leads to our earlier modified Cardy formula,

$$S_m = 2\pi \left(\sqrt{\frac{1}{6} c_L L_0^+} - \sqrt{\frac{1}{6} c_R L_0^-} \right). \quad (19)$$

Using this formula, a statistical interpretation of the entropy of exotic BTZ black holes is obtained in the dual CFT [10]. In the calculation here, we follow the presumption of Refs. [19,20] and regard the calculation about the part of τ

and $\bar{\tau}$ independently, that is the holomorphic factorization of the left and right sectors stated in Ref. [25]. Thus our results will not be plagued by the nonunitarity implied by the negative central charge. Moreover, mathematically the saddle approximation and the modular invariance required in the calculation above do not have the direct or intrinsic relation with the unitarity of the theory.

In particular, we can extend the calculation above and obtain a general formula in dual CFT for the purpose of explaining the entropy of BTZ black holes statistically,

$$S = \begin{cases} 2\pi \left(\sqrt{\frac{1}{6} c_L L_0^+} + \sqrt{\frac{1}{6} c_R L_0^-} \right), & c_L, c_R, L_0^+, L_0^- \geq 0 \\ 2\pi \left(\sqrt{\frac{1}{6} c_L L_0^+} - \sqrt{\frac{1}{6} c_R L_0^-} \right), & c_L, L_0^+ \geq 0, c_R, L_0^- \leq 0 \\ 2\pi \left(-\sqrt{\frac{1}{6} c_L L_0^+} + \sqrt{\frac{1}{6} c_R L_0^-} \right), & c_L, L_0^+ \leq 0, c_R, L_0^- \geq 0 \\ 2\pi \left(-\sqrt{\frac{1}{6} c_L L_0^+} - \sqrt{\frac{1}{6} c_R L_0^-} \right), & c_L, c_R, L_0^+, L_0^- \leq 0. \end{cases} \quad (20)$$

However, whether the last two formulas have an application to the statistical interpretation of the entropy of a 3D black hole is not clear up to now. In the next section, we will fill the implicit use of the third formula by finding a corresponding example in TMG.

III. STATISTICAL ENTROPY OF BTZ BLACK HOLES IN TMG

Start with the gravitational action of TMG [11],

$$I_T = \frac{1}{16\pi G} \int d^3 x \sqrt{-g} \left(R - 2\Lambda + \frac{1}{\mu} \mathcal{L}_{CS} \right), \quad (21)$$

where $\Lambda = -1/\ell^2$ is the negative cosmological constant and μ is the coupling parameter. It is easily seen that the TMG consists of normal Einstein gravity and LCS term, i.e., $\mathcal{L}_{CS} = \frac{1}{2} \epsilon^{\mu\nu\rho} \Gamma_{\mu\beta}^\alpha (\partial_\nu \Gamma_{\alpha\rho}^\beta + \frac{2}{3} \Gamma_{\nu\gamma}^\beta \Gamma_{\rho\alpha}^\gamma)$ where Γ is the Christoffel symbols. And the BTZ black-hole solution (1) satisfies the equations of motion of TMG,

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad (22)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the 3D Einstein tensor and $C_{\mu\nu} = \epsilon_\mu^{\tau\rho} \nabla_\tau (R_{\rho\nu} - \frac{1}{4} g_{\rho\nu} R)$ is the Cotton tensor. But the mass and the angular momentum of the black hole have the forms presented in Eq. (6), and in terms of the mass and angular momentum, the horizons change to

$$r_\pm = \sqrt{2G\ell} \left(a \sqrt{|M_T \ell + J_T|} \pm b \sqrt{|M_T \ell - J_T|} \right) \quad (23)$$

where $a = \frac{1}{\sqrt{|1+\frac{1}{\mu\ell}|}}$, $b = \frac{1}{\sqrt{|1-\frac{1}{\mu\ell}|}}$. Moreover, $M_T \ell - J_T = (1 - \frac{1}{\mu\ell})(m\ell - j)$ and $M_T \ell + J_T = (1 + \frac{1}{\mu\ell})(m\ell + j)$, and their signs depend on the value of the coupling parameter.

The central charges had been calculated in Ref. [15] using the Brown-Henneaux's canonical approach, i.e., $c_L = (1 + \frac{1}{\mu\ell})\frac{3\ell}{2G}$, $c_R = (1 - \frac{1}{\mu\ell})\frac{3\ell}{2G}$, and the zero modes L_0^+ and L_0^- of Virasoro algebras for the left and right movers are $L_0^+ = \frac{1}{2}(M_T\ell + J_T) = \frac{1}{2}(1 + \frac{1}{\mu\ell})(\ell m + j)$ and $L_0^- = \frac{1}{2}(M_T\ell - J_T) = \frac{1}{2}(1 - \frac{1}{\mu\ell})(\ell m - j)$. Then, according to our discussion about Cardy formula in the last section, we will calculate the entropy of BTZ black holes within the different parameter ranges.

- (i) when $\mu > 0$ and $\mu\ell > 1$, all central charges and the eigenvalues of the zero modes are positive, so its entropy is calculated as

$$\begin{aligned} S_T &= 2\pi \left(\sqrt{\frac{1}{6}c_L L_0^+} + \sqrt{\frac{1}{6}c_R L_0^-} \right) \\ &= \pi \sqrt{\frac{\ell}{2G}} \left(\left(1 + \frac{1}{\mu\ell}\right) \sqrt{\ell m + j} \right. \\ &\quad \left. + \left(1 - \frac{1}{\mu\ell}\right) \sqrt{\ell m - j} \right) \\ &= \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G}. \end{aligned} \quad (24)$$

This entropy is positive. In the case in which a lower bound of the mass is found, $M_T\ell - J_T \geq 0$ or $M_T \geq \frac{J_T}{\ell}$ which is similar to the situation of a BTZ black hole in the normal Einstein gravity. In particular, it is noted that the case is just that discussed in the Ref. [15]. Moreover, for the case $\mu < 0$ and $\mu\ell < -1$, all central charges and the eigenvalues of the zero modes are also positive, so its entropy is still $S_T = \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G}$. In particular, when $\mu\ell = \pm 1$, one of the central charges disappears, which leads to the chiral gravity [26], but their BTZ black-hole entropies still conform to the above calculation, i.e., $S_T = \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G} = \frac{\pi r_+}{2G} \pm \frac{\pi r_-}{2G}$.

- (ii) when $\mu > 0$ and $0 < \mu\ell < 1$, we have $c_R < 0$ and $L_0^- < 0$, so its entropy should be calculated according to

$$\begin{aligned} S_T &= 2\pi \left(\sqrt{\frac{1}{6}c_L L_0^+} - \sqrt{\frac{1}{6}c_R L_0^-} \right) \\ &= \pi \sqrt{\frac{\ell}{2G}} \left(\left(1 + \frac{1}{\mu\ell}\right) \sqrt{\ell m + j} \right. \\ &\quad \left. - \left(\frac{1}{\mu\ell} - 1\right) \sqrt{\ell m - j} \right) \\ &= \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G}. \end{aligned} \quad (25)$$

This entropy is also positive. In the case in which an upper bound of the mass is found, $M_T\ell - J_T \leq 0$ or

$M_T \leq \frac{J_T}{\ell}$ which is similar to the situation of an exotic BTZ black hole.

- (iii) when $\mu < 0$ and $-1 < \mu\ell < 0$, we have $c_L < 0$ and $L_0^+ < 0$, so its entropy should be calculated according to

$$\begin{aligned} S_T &= 2\pi \left(-\sqrt{\frac{1}{6}c_L L_0^+} + \sqrt{\frac{1}{6}c_R L_0^-} \right) \\ &= \pi \sqrt{\frac{\ell}{2G}} \left(-\left|1 + \frac{1}{\mu\ell}\right| \sqrt{\ell m + j} \right. \\ &\quad \left. + \left(1 - \frac{1}{\mu\ell}\right) \sqrt{\ell m - j} \right) \\ &= \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G}. \end{aligned} \quad (26)$$

where it is noted that the positivity of the entropy requires $-1 < \mu\ell \leq -\frac{r_-}{r_+}$, because, for the micro-canonical ensemble, the negative entropy means the number of microstates in the CFT is less than 1, which is hard to accept physically. Moreover, it is interesting to note that the excluded range could be expressed as $-\frac{r_-}{r_+} < \mu\ell < 0$ or $-1 < \frac{\mu}{\Omega} < 0$ where $\Omega = \frac{r_-}{\ell r_+}$ is the angular velocity of BTZ black holes, and so in order to preserve the positive entropy, the coupling strength $|\mu|$ cannot be smaller than the angular velocity Ω . On the other hand, we note that the excluded range $-\frac{r_-}{r_+} < \mu\ell < 0$ leads to the negative mass of BTZ black holes in TMG, i.e., $\ell M_T =$

$$\begin{aligned} \ell m + \frac{1}{\mu\ell} j &< \ell m - \frac{r_+}{r_-} j = \ell m - \frac{\sqrt{\ell m + j} + \sqrt{\ell m - j}}{\sqrt{\ell m + j} - \sqrt{\ell m - j}} j = \\ &\ell m - \frac{\ell m + \sqrt{\ell m + j} \sqrt{\ell m - j}}{j} j = -\sqrt{\ell m + j} \sqrt{\ell m - j} \leq \\ &0. \end{aligned}$$

But the linearized analysis gave the classical stability of the BTZ black hole in TMG for all the values of the coupling parameter [27]. This is peculiar for a stable black hole with a negative mass and so it deserves further investigation for the stability of the BTZ black hole in the background of TMG. Moreover, if the conclusion of Ref. [27] is true, we could guess that the negative sign of the entropy in the range $-\frac{r_-}{r_+} < \mu\ell < 0$ might only be due to the negative sign of the mass. Since the black hole in the range $-\frac{r_-}{r_+} < \mu\ell < 0$ could still be stable, the corresponding number of its microscopical states might be suggested accordingly as $e^{|S_T|}$, which remains to be confirmed in the future quantum gravity theory.

Thus, based on our analysis using the Cardy formula, the counting of the microstates gives the same form of the entropy,

$$S_T = \frac{\pi r_+}{2G} + \frac{1}{\mu\ell} \frac{\pi r_-}{2G}, \quad (27)$$

which is different from the suggested results in Ref. [17] but consistent with many other methods [12,13,16], and the entropy is positive except in case (iii) the more tightened condition is required. Moreover, the cases (i)–(iii) have not included the use of the last formula of the general Cardy formula (20), and so it is still unclear for its possible function as a statistical formula for some black-hole entropies. To some extent, this also provided an interesting motivation to find a 3D gravity theory with both the left and right negative central charges.

Then we have to investigate whether the entropy (27) obtained statistically satisfies the laws of thermodynamics. First the entropy satisfies the first law

$$dM_T = T_H dS_T + \Omega dJ_T \quad (28)$$

with the thermodynamic variables $T_H = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell^2}$, $\Omega = \frac{r_-}{\ell r_+}$ which are geometrical in the sense that they depend only on the location of the Killing horizons and are model independent, i.e., independent of the specific field equations that are solved by the BTZ metric. Remarkably, in our earlier paper [10], it was pointed out that for the forms (6) of the mass and the angular momentum, only the entropy (27) obtained here can preserve the first law. Thus our calculation here confirms that the entropy obtained by a statistical method is consistent with the requirement of black-hole thermodynamic laws.

Once the first law is held, it is easier to confirm the second law by the so-called method of ‘‘physical process’’ [28], i.e., $dS_T = \frac{1}{T_H}(dM_T - \Omega dJ_T) = \frac{2}{T_H} dM_T > 0$ for the process of the particle absorption by a black hole, where the relations $\frac{dM_T}{dJ_T} = \frac{dM_T}{dS_T} \frac{dS_T}{dJ_T} = -\Omega$ and $S_T = S_T(M_T, J_T)$ are used. In particular, for BTZ black holes in any 3D gravity theory, the particle absorption generally leads to the change of the entropy $dS_T = \frac{2}{T_H} dM_T$, which is similar to the situation of the Schwarzschild black hole in four-dimensional spacetime, for which the change of the entropy is proportional to the term $\frac{\Delta E}{M}$, where ΔE is the energy of absorbed particles. Of course that is only valid for the quasistationary situation, but a general analysis of the particle absorbing process had been made in a recent work [29], and it is easy to apply and extend their conclusions to the situation of TMG.

IV. LOGARITHMIC CORRECTION

From the above discussion, we knew that the LCS term gave a BTZ black-hole entropy associated with the outer (event) horizon a term proportional to the length of the inner horizon. The peculiar behavior made the author of Ref. [13] speculate that the observer at infinity might see the interior of the black hole when the LCS term was included. Thus it is interesting to explore whether the influence of the LCS term could extend to the logarithmic correction of the black-hole entropy associated with the

outer horizon, since the logarithmic correction is usually believed to be caused by the quantum gravity effect [30,31] in the calculation using the methods of string theory [32] and loop quantum gravity [33].

From the density of states presented in Eq. (18), one can get the logarithmic correction of the entropy associated with the outer horizon by the part before exponential term,

$$\Delta S = \frac{1}{4} \ln \left(\frac{c_{LCR}}{96^2 (L_0^+ L_0^-)^3} \right). \quad (29)$$

For the normal 3D Einstein gravity, it had been calculated [20,21] as

$$\Delta S_G = -\frac{3}{2} \ln \frac{\pi r_+}{2G} - \frac{3}{2} \ln \kappa \ell - \frac{3}{2} \ln \frac{\pi \ell}{2G} - \ln 8G\ell^2, \quad (30)$$

where $\kappa = \frac{r_+^2 - r_-^2}{r_+ \ell^2}$ is the surface gravity of the outer horizon, and it is also geometric and a constant for a specific BTZ black hole. For the exotic 3D Einstein gravity, the logarithmic correction is the same as in Eq. (30) up to a different constant term.

Then for TMG, the correction of the entropy is

$$\Delta S_T = -\frac{3}{2} \ln \frac{\pi r_+}{2G} - \frac{3}{2} \ln \kappa \ell - \frac{3}{2} \ln \frac{\pi \ell}{2G} - \ln 8G\ell^2 ab, \quad (31)$$

where a and b , the same as that presented in Eq. (23), are related to the coupling parameter μ . Comparing Eqs. (30) and (31), we find that the difference exists only in the last constant term. This shows that even in the gravity theories with the LCS term, the logarithmic correction of the entropy associated with the outer horizon is not related to the length of the inner horizon. Thus it hints that the influence of LCS term on the entropy of the black hole only works for the leading order term and cannot be extended into the deeper knowledge of the black-hole entropy, i.e., logarithmic correction.

V. DISCUSSION AND CONCLUSION

In the paper, we have investigated the Cardy formula and have made the calculation for the case with the negative central charge. Our results showed that in the assumption of holomorphic factorization, for the same BTZ black-hole metric form and the same asymptotic AdS condition, the Cardy formula is *not* completely the same and depends on the sign of the central charges and zero eigenvalues of the left and right movers \hat{L}_m^\pm . This validated and extended our earlier result [10]. We have also applied these results to the BTZ black holes in the TMG and obtained the same entropy as that by other different methods. Thus we provided a statistical interpretation again for a non-Bekenstein-Hawking entropy. In particular, different from the exotic Einstein gravity, the BTZ black-hole entropy in TMG has to be explained statistically through different Cardy formulas in the different coupling parameter ranges.

It is also noted that the statistical entropy of BTZ black holes in higher curvature gravity had been calculated [34], and when the LCS term was added to the higher curvature gravity, e.g., general massive gravity theory [35], the calculation of the leading term by the Cardy formula still gives a non-Bekenstein-Hawking entropy statistically. After calculating the black-hole entropy statistically in TMG, we have also discussed the thermodynamic first and second laws, and found that they both apply for such entropy. Thus the obtained statistical entropy is still thermodynamic entropy, although it does not take the Bekenstein-Hawking form.

It is noted that in all the three cases involved, i.e., the normal and exotic 3D Einstein gravity and TMG, there is an obvious difference which is reflected on the parity of the respective bulk theories of gravity. Although it is still unclear for the role of parity in the statistical interpretation of the black-hole entropy, the advantage of definite parity in calculating the black-hole entropy using the method of canonical quantization has been discussed [36]. Thus what role the parity of the general gravity theory would play and how it would play in the process of calculating the corresponding black-hole entropy deserves further investigation.

The derivation of non-Bekenstein-Hawking entropy is due to the introduction of the LCS term which might be the reason for scrambling the parity in TMG. From our analysis, although the LCS term does not influence the thermodynamics of the outer horizon, it indeed gives a peculiar term which is proportional to the length of the inner horizon in the expression of the black-hole entropy of the outer horizon. We have investigated whether the LCS term influenced the term of logarithmic corrections to the black-hole entropy associated with the outer horizon, and found the answer to be no, any influences existed up to a constant term. We concluded that the influence of the LCS term will not extend to the deeper level of the black-hole entropy at least in 3D spacetime, which will be significant for understanding exactly the role of LCS term in the general 3D gravity or quantum gravity theories.

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